Wave Mechanics and Large-scale Structure

Peter Coles
Cardiff University
(@telescoper)

Structure formation basics

- Small primordial density perturbations grow via the mechanism of gravitational instability.
- Large density fluctuations observable today thought to be dominated by non-baryonic matter.
- Observations of clustering support some form of collisionless CDM.
- Approximate model of structure formation:

> *Large-scale structure is the result of the gravitational amplification of small inhomogeneities in the primordial CDM distribution.*

The fluid approach

- Treat collisionless CDM as a fluid.
- Linear perturbation theory gives an equation for the *density contrast* $\delta = \rho / \rho_b - 1$
- In a spatially flat CDM-dominated universe $\delta(\mathrm{X}, a) = D_{\pm}(a)\delta_i(\mathrm{X})$ where:

$$D_+(a) \propto a \qquad \text{Growing mode}$$

$$D_-(a) \propto a^{-3/2} \qquad \text{Decaying mode}$$

- Comoving velocity $\mathrm{U} = d\mathrm{X} / da$ associated with the growing mode is *irrotational*:

$$\mathrm{U} = -\nabla_{\mathrm{X}} \phi_U$$

Problems with the fluid approach

- Linear theory only valid at early times when fluctuations in physical fluid quantities are small.

- Perturbations grow and the system becomes non-linear in nature.

- Linear theory predicts the existence of spatial regions with negative density $\delta < -1$!

The Zel’dovich approximation

- Follows perturbations in particle trajectories:

$$X(Q,a) = Q + aU(Q)$$

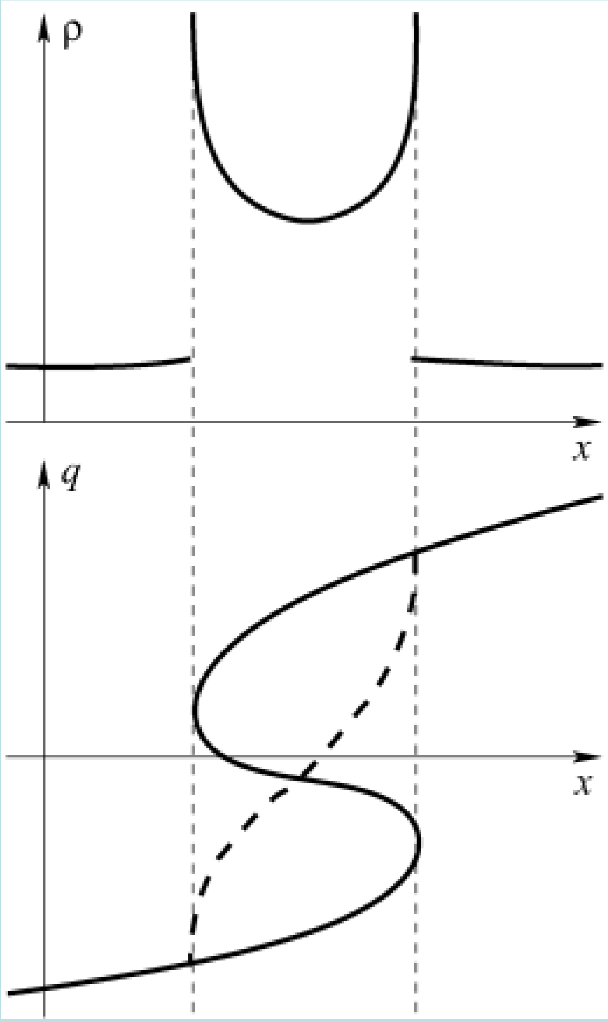
- Mass conservation leads to:

$$\delta(X,a) = \frac{1}{\prod_{i=1}^{3}(1 - a\lambda_i)} - 1$$

- Zel’dovich approximation remains valid in the quasi-linear regime, after the breakdown of the linearised fluid approach.

Problems with the Zel'dovich approximation

- The Zel'dovich approximation fails when particle trajectories cross – *shell crossing.*
- A region where shell-crossing occurs is called a *caustic*.
- At caustics the mapping $\mathrm{Q} \mapsto \mathrm{X}$ is no longer unique and the density becomes infinite.
- Particles pass through caustics without responding to the large gravitational force $\Longrightarrow$ non-linear regime described very poorly.



https://www.youtube.com/watch?v=0fjk8X1KuyE

The wave-mechanical approach

- Assume the comoving velocity is irrotational: $\mathrm{U} = -\nabla_{\mathrm{X}} \phi_U$
- The equations of motion for a fluid of gravitating CDM particles in an expanding universe are then:

$$\frac{\partial \phi_U}{\partial a} - \frac{1}{2}(\nabla_{\mathrm{X}} \phi_U)^2 - V = 0 \qquad \text{Bernoulli}$$

$$\frac{\partial \eta}{\partial a} - \nabla_{\mathrm{x}} \cdot (\eta \nabla_{\mathrm{X}} \phi_U) = 0 \qquad \text{Continuity}$$

where $\eta = \rho / \rho_b$ and

$$V = \frac{\Phi_p}{a^2 \dot{a}^2} - \left(\frac{2}{a} + \frac{\ddot{a}}{\dot{a}^2} \right) \phi_U \qquad \text{‘Modified potential’}$$

The wave-mechanical approach

- Apply the *Madelung transformation* $\psi = \sqrt{\eta}\exp(-i\phi_U / \upsilon)$ to the fluid equations.
- Obtain the Schrodinger equation:

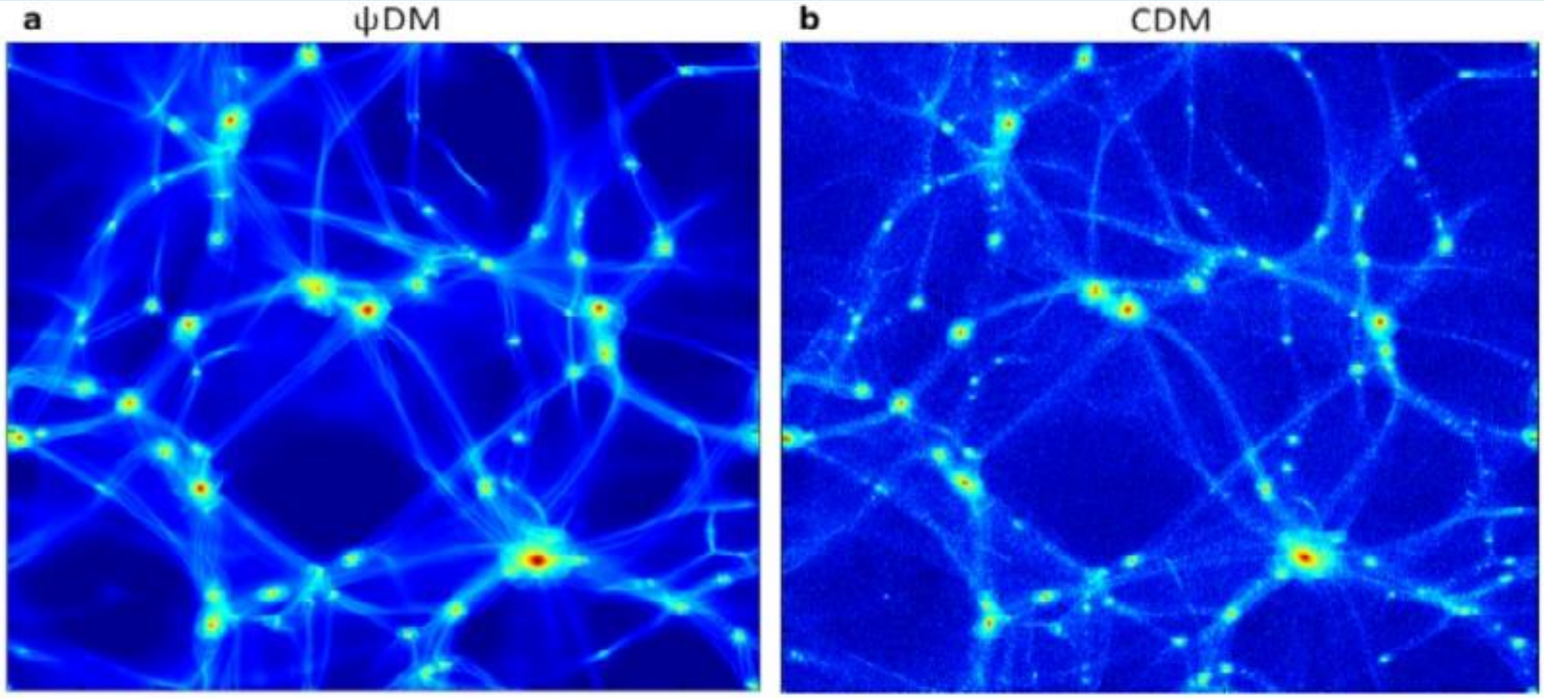
$$i\upsilon\frac{\partial\psi}{\partial a} = \left[-\frac{\upsilon^2}{2}\nabla_x^2 + V + P\right]\psi$$

- $P = \frac{\upsilon^2}{2}\frac{\nabla_x^2\sqrt{\eta}}{\sqrt{\eta}}$ is the *quantum pressure* term.
- DeBroglie wavelength $\lambda_{dB} \propto \upsilon$

Might dark matter *really* be quantum-mechanical?

- There is evidence that CDM does not fit on small scales: dwarf galaxies, `cuspy' cores, etc..
- Simple idea): DM is a (very) light particle (m~ 10^{-27} eV) then the Compton wavelength can be a galactic scale.
- In this case the `quantum pressure' is a real physical effect.
- Something like `warm' dark matter arises (actually `fuzzy' dark matter)

From Schive et al., arXiv: 1406.6586 (also published in *Nature*)



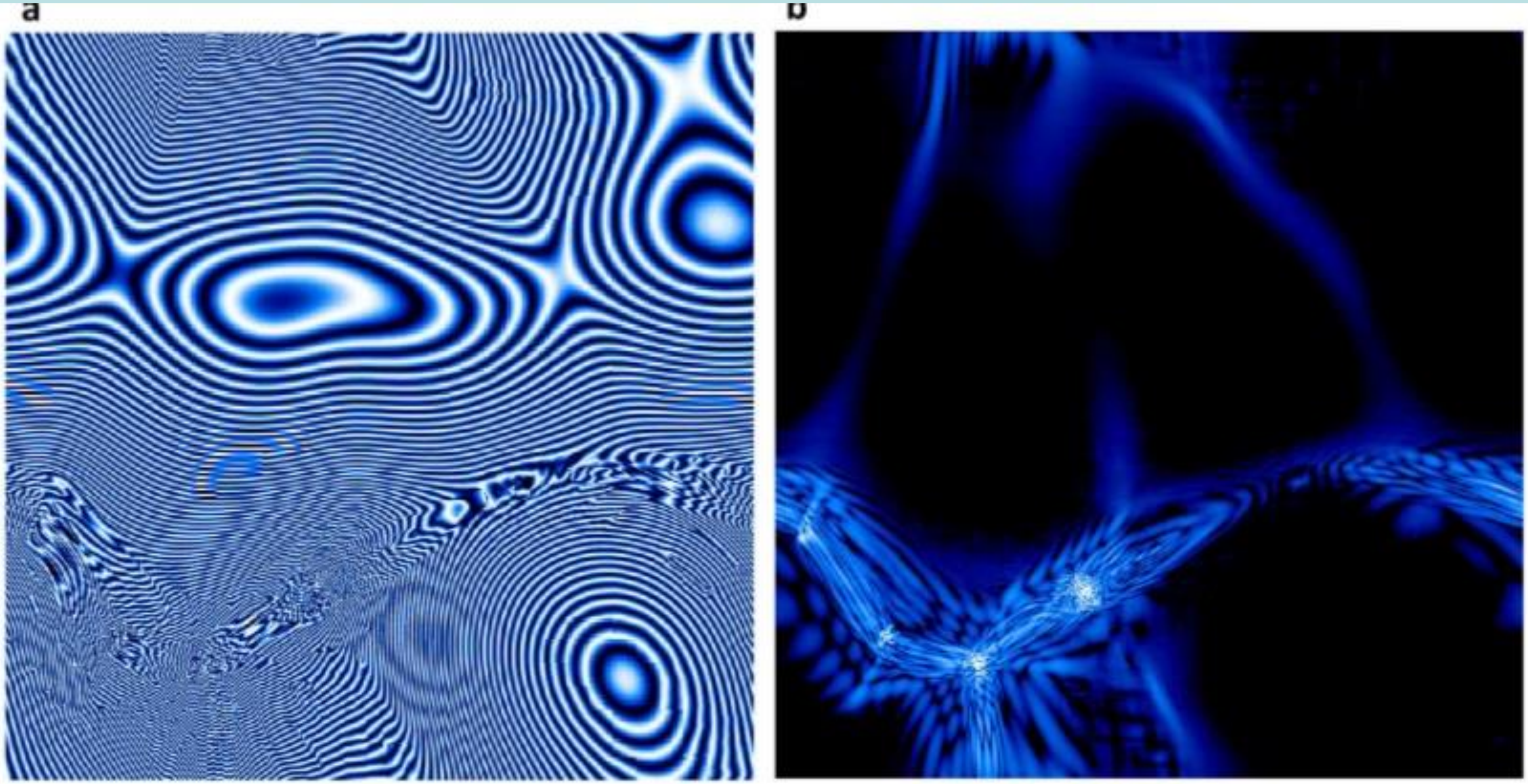
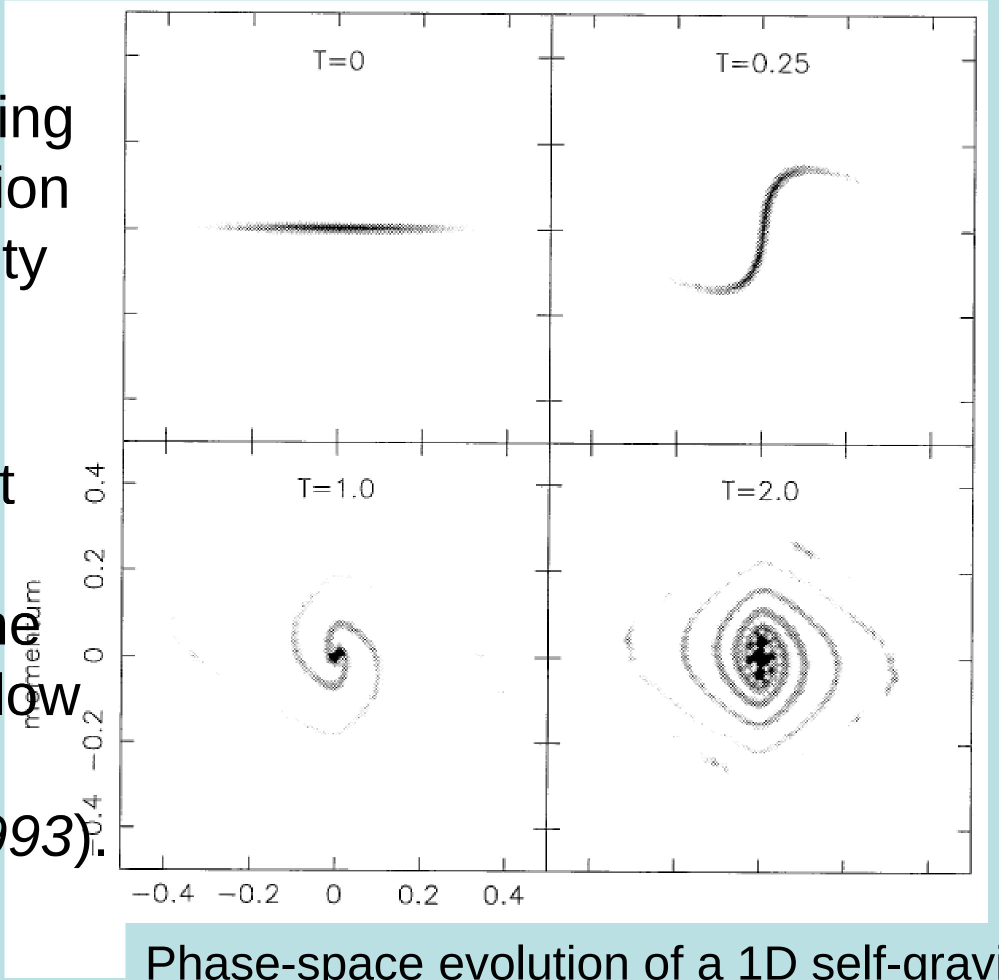


Figure S1: Square wave function ψ^2 ($\equiv f^2 e^{i2S}$) in the ψDM simulation. Panels (**a**) and (**b**) show a 2 Mpc slice of phase ($sin(2S)$) and amplitude (f^2) of the wave function at $z = 3.1$, respectively. The simulation challenge arises from the complexity of the wave function. Strong and rapid phase oscillations are common everywhere (even in the low-density background shown by the dark regions in the density plot), where sufficient spatial and temporal resolution is required to resolve each wavelength.

The wave-mechanical approach

- For a collisionless medium, shell-crossing leads to the generation of vorticity $\Longrightarrow$ velocity flow no longer irrotational!
- Possible to construct more sophisticated representations of the wavefunction that allow for multi-streaming (*Widrow & Kaiser 1993*).



Phase-space evolution of a 1D self-gravitating system with $\rho_i(X) = \rho_0 \exp(-X^2/L^2)$, $v_i(X) = 0$

The ‘free-particle’ Schrodinger equation

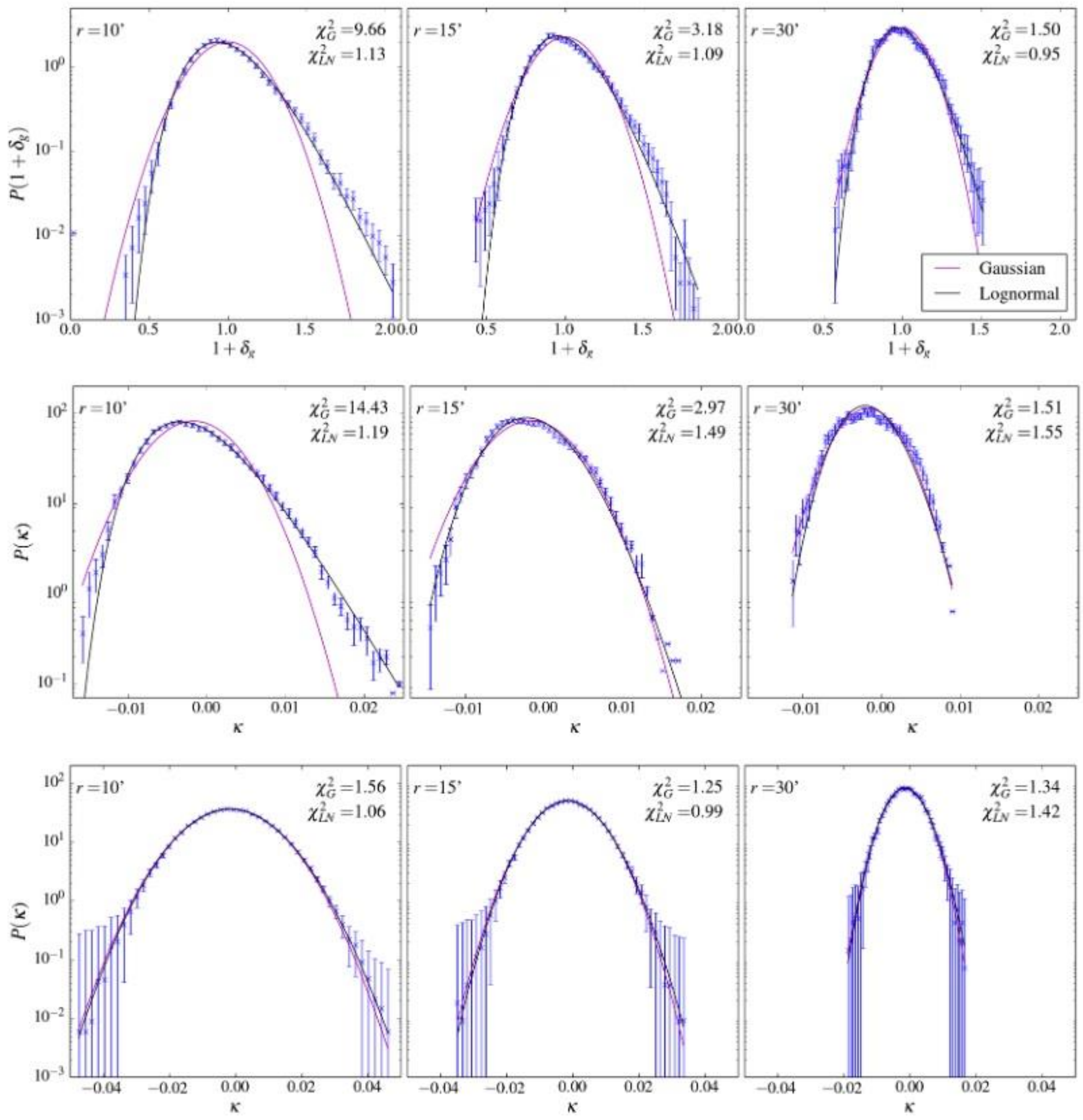
- In a spatially flat CDM-dominated universe, the ‘potential’ $V \equiv 0$ in the linear regime.

- Neglecting quantum pressure, the Schrodinger equation to be solved is then the ‘free-particle’ equation:

$$i\upsilon \frac{\partial \psi}{\partial a} = -\frac{\upsilon^2}{2} \nabla_{\mathrm{x}}^2 \psi$$

- Can be solved exactly!

MICE



r = 10'
χ^2_G = 9.66
χ^2_{LN} = 1.13
r = 15'
χ^2_G = 3.18
χ^2_{LN} = 1.09
r = 30'
χ^2_G = 1.50
χ^2_{LN} = 0.95
Gaussian
Lognormal
$P(1+\delta_R)$
$1+\delta_R$
r = 10'
χ^2_G = 14.43
χ^2_{LN} = 1.19
r = 15'
χ^2_G = 2.97
χ^2_{LN} = 1.49
r = 30'
χ^2_G = 1.51
χ^2_{LN} = 1.55
$P(\kappa)$
κ
r = 10'
χ^2_G = 1.56
χ^2_{LN} = 1.06
r = 15'
χ^2_G = 1.25
χ^2_{LN} = 0.99
r = 30'
χ^2_G = 1.34
χ^2_{LN} = 1.42
$P(\kappa)$
κ

Mon. Not. R. astr. Soc. (1991) **248**, 1–13

A lognormal model for the cosmological mass distribution

Peter Coles[1]★ and Bernard Jones[2]†
[1]*Astronomy Centre, University of Sussex, Falmer, Brighton BN1 9QH*
[2]*Niels Bohr Institute, Blegdamsvej 17, DK-2100 Copenhagen 0, Denmark*

Accepted 1990 June 18. Received 1990 June 18; in original form 1990 May 14

SUMMARY

We discuss the use of a lognormal (LN) random field as a model for the distribution of matter in the Universe. We find a number of reasons why this should be a plausible approximation to the distribution of density irregularities obtained by evolving from Gaussian initial conditions. Unlike straightforward linear theory, the model always has $\rho > 0$ but is arbitrarily close to the Gaussian at early times. It has the added advantage that, like the Gaussian model, all its statistical properties can be formulated in terms of one covariance function.

A number of interesting and important difficulties with the statistical treatment of density perturbations are revealed by an analysis of this model. In particular, the LN model is not completely specified by its moments. We explain why this could be true for the actual matter field. We also show that the usual method of representing the three- and four-point correlation functions of galaxies, in terms of the parameters Q and R, is not useful for discriminating between Gaussian and non-Gaussian fluctuations, and propose better parameterizations in terms of the skewness and kurtosis of the three- and four-point distributions, respectively.

Other characteristics of the model, such as topology (genus curves, etc.), multifractal behaviour, void probabilities and biasing (behaviour of 'peaks' relative to background fluctuations) are also discussed. The model also provides a way of checking the consistency of treatments of large-scale streaming motions in the Universe by allowing us to determine the scale at which linear theory cannot be accurate for both the matter and velocity fields.

We discuss a possible model for the number-count distribution of galaxies, based on the LN distribution but allowing for discreteness effects which can make the distribution of log n appear non-Gaussian, and show how to construct Monte-Carlo simulations of point patterns (in one-, two, or three-dimensions) which contain correlations of all orders.

1 INTRODUCTION

Ever since the pioneering studies of Neyman & Scott (1952) it has been understood that our knowledge of the distribution of galaxies is statistical and, therefore, incomplete in the sense that we will never be able to predict the specific locations of galaxies around us. It is also true that our knowledge is incomplete even in a statistical sense: the distribution of galaxies in space is only completely specified by the hierarchy‡ of n-dimensional joint probability density functions, $f_n(\rho)$, connecting the density at different spatial positions and our knowledge of these is restricted to low-order moments and related functions. Just as the analysis of galaxy catalogues gives us only partial information about the f_n, so is it true that physical models do not allow the f_n to be expressed in any analytic form. The late stages of galaxy formation are usually modelled by discrete numerical simulations (Efstathiou *et al.* 1985, and references therein) and one cannot extract any more information about the f_n from these than one could from a galaxy catalogue. Analytic approximations for the growth of non-linear structure such as the Zel'dovich approximation (Zel'dovich 1970; Bond & Couchman 1988; Shandarin & Zel'dovich 1989), second-order perturbation theory (Juszkiewicz, Sonoda & Barrow 1984; Coles 1990) or those techniques based upon the

★Present address: Astronomy Unit, Queen Mary and Westfield College, Mile End Road, London E1 4NS.
†Present address: Astronomy Centre, University of Sussex, Falmer, Brighton BN1 9QH.
‡Doob (1953) first demonstrated this fact. Doob defines two stochastic processes to be *stochastically equivalent* only if all finite dimensional joint distributions are identically equal for the two distributions.

Gravitational collapse in one dimension

- Assume a sinusoidal initial density profile in 1D:

$$\delta_i(X) = \delta_0 \cos\left(\frac{2\pi X}{D}\right)$$

where D is the comoving period of the perturbation.

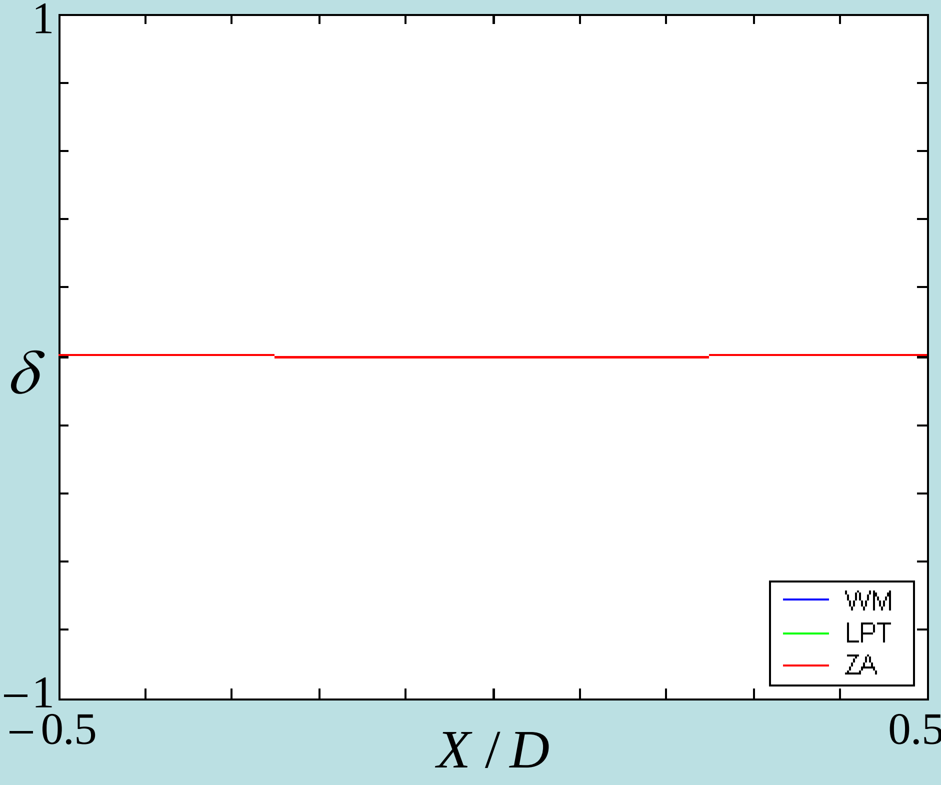
- Free parameters are:

1. *The amplitude* δ_0 *of the initial density fluctuation.*
2. *The dimensionless number* $R_e = D^2 / a_i \upsilon$

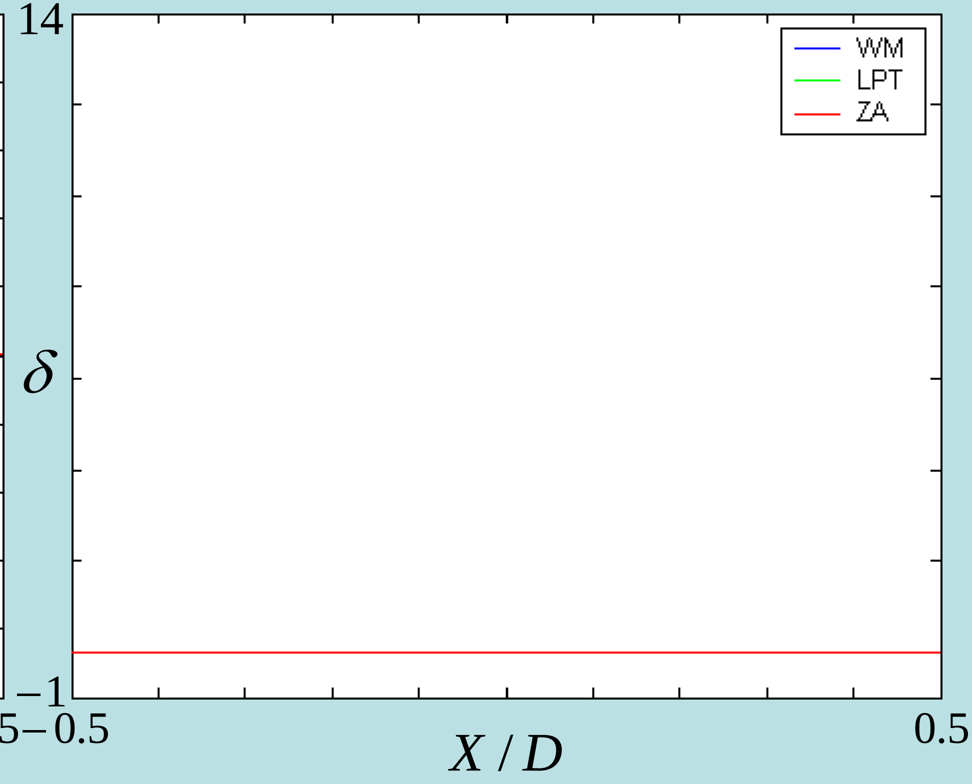
- Quantum pressure $P \propto 1/R_e^2$
- DeBroglie wavelength $\lambda_{dB} \propto 1/R_e$

Gravitational collapse in one dimension

$\delta_0 = -0.001$
$R_e = 1.0 \times 10^3$



$\delta_0 = -0.001$
$R_e = 1.2 \times 10^7$



Evolution of a periodic 1D self-gravitating system with $\delta_i(X) = \delta_0 \cos(2\pi X / D)$

Cosmic reconstruction

- Gravity is invariant under time-reversal!
- The reconstruction question:

> *Given the large-scale structure observable today, can we reverse the effects of gravity and recover information about the universe at* $z \sim 1000$*?*

- Non-linear gravitational evolution from $z \sim 10$ is the main obstacle to reconstruction.
- Non-linear multi-stream regions prevent unique reconstruction.
- At scales above a few Mpc multi-streaming is insignificant

$\Longrightarrow$ smoothing necessary.

The Zel'dovich-Bernoulli method

- In Eulerian space the Zel'dovich approximation becomes:

$$\frac{\partial \phi_U}{\partial a} - \frac{1}{2}(\nabla_x \phi_U)^2 = 0 \qquad \text{Zel'dovich-Bernoulli}$$

$$\phi_U = \frac{\Phi_p}{(2a\dot{a}^2 + a^2\ddot{a})}$$

- Reconstruction process:

1. *Determine present comoving velocity potential* $\phi_{U,0}$
2. *Smooth to remove non-linearities.*
3. *Integrate ZB equation backwards from* $a_0 = 1$ *to* $a_i \sim 0.001$
4. *Use linear theory to calculate initial density field* δ_i

Wave-mechanics and the Zel'dovich-Bernoulli method

- The Zel'dovich-Bernoulli equation can be replaced by the 'free-particle' Schrodinger equation!
- Currently testing the `free-particle' reconstruction method on a 2D N-body simulation.
- If successful, possible extensions are:

1. *Model errors in galaxy position and velocity measurements by exploiting the nature of quantum mechanics.*
2. *Work in redshift space coordinates* $\mathrm{S} = \mathrm{X} + a\hat{\mathrm{S}}(\mathrm{U} \cdot \hat{\mathrm{S}})$
3. *Generalise to 3D.*

Summary

- The wave-mechanical approach can overcome some of the main difficulties associated with the fluid approach and the Zel’dovich approximation.
- More sophisticated representations of the wavefunction can be used to allow for multi-streaming; the quantum pressure term is crucial in determining how well the wave-mechanical approach performs.
- The `free-particle’ Schrodinger equation can be applied to the problem of reconstruction.
- Dark Matter may even be quantum-mechanical!